# Indian Institute of Technology Kanpur

## Department of Mathematics and Statistics

### Admission Test for PhD in Mathematics (2023-I)

**Date**: 10 May 2023

Time: 2 hours

Total marks: 80

# Instructions

1. This question cum answer booklet has two parts viz. Part-A and Part-B. Part-A consists of 8 multiple answer type questions and Part-B contains 4 short answer type questions. All the questions in both the sections are compulsory.

2. Questions in Part-A may have more than one correct answer (option). Five (**5**) marks will be awarded if you choose **all** the correct option(s) and do not choose any **wrong** option(s). Three (**3**) marks will be awarded if you choose **at least one** correct option, but **not all** the correct options, and **no wrong** option. In all other cases, zero marks will be awarded.

3. Answers to questions in Part-A must be filled in the **bubble sheet** provided. Answers to each question in Part-B **must be written** within the space provided for the same. **No additional sheet(s)** will be provided.

4. Please write your full name and application number **legibly** on each page, including the bubble sheet, in the space provided for this purpose. **Pages which do not contain your name and application number will not be graded.**

5. Each of the four questions in Part-B is for ten (**10**) marks. The answer to each of these questions must contain all the necessary and relevant details. Marks will be deducted for missing details/steps.

6. This booklet must be returned to the invigilator before leaving the examination hall.

## **Part-A:** Multiple selective questions

1. Let $(\mathscr{H}, \langle\, ,\, \rangle)$ be a Hilbert space and $T : \mathscr{H} \longrightarrow \mathscr{H}$ be a linear map satisfying

$$\langle Tx, y\rangle = \langle x, Ty\rangle,\ \forall x, y \in \mathscr{H}. \tag{1.1}$$

Then, which of the following statements is(are) **true**?

(a) If $T$ is bounded then it satisfies equation (1.1).

(b) If $T$ satisfies equation (1.1) then it is bounded.

(c) If $B \stackrel{\text{def}}{=} \{x \in \mathscr{H} : \langle x, x\rangle \leq 1\}$ denotes the closed unit ball in $\mathscr{H}$, then $\overline{T(B)}$ is compact.

(d) $\langle T^2x, x\rangle > 0$, for all $x \in \mathscr{H} \setminus \{0\}$.

2. Let $\Omega$ be an open connected set in $\mathbb{C}$ such that for each $z \in \Omega$, $\bar{z} \in \Omega$. Consider a holomorphic function $f : \Omega \longrightarrow \mathbb{C}$ satisfying $f(\Omega \cap \mathbb{R}) \subset \mathbb{R}$. Then, which of the following statements is(are) **true**?

   (a) $f'(\Omega \cap \mathbb{R}) \subset \mathbb{R}$.

   (b) $f(\bar{z}) = f(z)$ for all $z \in \Omega$.

   (c) $f(\bar{z}) = \overline{f(z)}$ for all $z \in \Omega$.

   (d) $f'(\bar{z}) = f'(z)$ for all $z \in \Omega$.

3. For any two vectors $v, w \in \mathbb{R}^n$, the standard inner product of $v$ and $w$ is denoted by $\langle v, w\rangle$. Let $S$ be a subset of unit vectors in $\mathbb{R}^n$ such that for all $v, w \in S$ with $v \neq w$ we have $\langle v, w\rangle < 0$. Then, which of the following statements is(are) **true**?

   (a) $S$ is a finite set.

   (b) $S$ is a linearly independent subset of $\mathbb{R}^n$.

   (c) $|S| < n + 1$, where $|S|$ stands for the cardinaility of $S$.

   (d) $|S| \leq n + 1$.

4. Which of the following statements is(are) **true**?

   (a) Only one of the polynomials $x^3 - 2$ and $x^3 - 3$ is irreducible in

   $$\mathbb{Q}(i) \stackrel{\text{def}}{=} \{a + b\sqrt{-1} : a, b \in \mathbb{Q}\}.$$

   (b) The ideal generated by 7 in the ring $\mathbb{Z}[i] \stackrel{\text{def}}{=} \{a + b\sqrt{-1} : a, b \in \mathbb{Z}\}$ is a maximal ideal.

   (c) The ideal $(2x^2 - 1)$ is a prime ideal in the polynomial ring $\mathbb{Z}[x]$

   (d) The group of units in the ring $\mathbb{Z}[\sqrt{5}] \stackrel{\text{def}}{=} \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$ is finite.

5. Which of the following statements is(are) **true**?

   (a) If $X$ is a complete metric space and $Y$ is a closed subset of $X$ then $Y$ is compact.

   (b) $\left\{\left(x, \sin\frac{1}{x}\right) : x > 0\right\} \cup \{(0, 0)\}$ is closed in $\mathbb{R}^2$.

   (c) $\mathbb{R}^3 \setminus \mathbb{Q}^3$ is connected.

   (d) Consider the projection map $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $\pi(x, y) = x$. If $E$ is closed subset of $\mathbb{R}^2$ then $\pi(E)$ is closed in $\mathbb{R}$.

6. Let $X$ and $Y$ be Hausdorff topological spaces and $f : X \longrightarrow Y$ be a continuous map. Then, which of the following statements is(are) **true**?

   (a) If $X$ is compact and $f$ is a bijection then $f$ is a homeomorphism.

   (b) If $X$ is a complete metric space and $f$ is a bijection then $f$ is a homeomorphism.

   (c) If $f$ is onto then there exists a continuous map $g : Y \longrightarrow X$ such that $g \circ f$ is the identity mapping on $X$.

   (d) If $E$ is a closed subset of $X$ then $f(E)$ is a closed subset of $Y$.

7. Let $y_0 \in \mathbb{R}$ and consider the following initial value problem (IVP):

$$y' = -y^3(1 + \sin(y)) \text{ and } y(0) = y_0.$$

   Then, which of the following statements is(are) **true**?

   (a) At least two solutions of the IVP exist in a neighbourhood of 0.

   (b) The IVP has exactly one solution in a neighbourhood of 0.

   (c) The IVP has no solution in any neighbourhood of 0 when $y_0 \neq 0$.

   (d) The IVP has exactly one solution in $(-\infty, \infty)$.

8. Let $u$ satisfy the initial value problem $u_y(x, y) + u(x, y)u_x(x, y) = 0$ in $\mathbb{R}^2$ with the initial data $u(x, 0) = \phi(x)$, where $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function such that $c \stackrel{\text{def}}{=} \phi(0) - \phi(1) > 0$. Then, which of the following statements is(are) **true**?

   (a) $u(0, 0) = u(1, 0)$.

   (b) $u(0, 0) = u\left(\frac{\phi(0)}{c}, \frac{1}{c}\right)$.

   (c) $u(1, 0) = u\left(\frac{\phi(0)}{c}, \frac{1}{c}\right)$.

   (d) The line $\{(x, 0)\}$ is non-characteristic with respect to the given PDE.

   (e) The line $\{(0, y)\}$ is never a characteristic curve of the given PDE.

## **Part-B:** Short answer type questions

| Name: | Marks obtained ↓ |
|---|---|
| Application no.: IITKPG1/Ph.D/MTH/231000 ____ | |

1. Let $\gamma : [0,1] \longrightarrow \mathbb{C}$ be a continuous curve and $f : [0,1] \longrightarrow \mathbb{R}$ be Riemann integrable. Prove or disprove the following:

$$F(z) \stackrel{\text{def}}{=} \int_0^1 \frac{f(t)}{\gamma(t) - z}\, dt,\ \forall z \notin \gamma([0,1]),\ \text{is analytic.}$$

[10]

| Name: | Marks obtained ↓ |
|---|---|
| Application no.: IITKPG1/Ph.D/MTH/231000 ____ | |

2. Prove or disprove that every group of order 2023 is abelian. [10]

| Name: | Marks obtained ↓ |
|---|---|
| Application no.: IITKPG1/Ph.D/MTH/231000 ____ | |

3. Show that every compact metric space has a countable dense subset. [10]

| Name: | Marks obtained ↓ |
|---|---|
| Application no.: IITKPG1/Ph.D/MTH/231000 ____ | |

4. Let $\alpha : (a, b) \longrightarrow \mathbb{R}$ be a continuous function. Consider the following ordinary differential equation (ODE):

$$y'' + \alpha y = 0.$$

   (a) Let $y$ be a nontrivial solution of the given ODE. Show that, if $y(t_0) = 0$, where $t_0 \in (a, b)$, then $y'(t_0) \neq 0$.

   (b) Show that the zeros of a nontrivial solution of the given ODE are *isolated*, i.e., if $y$ is a nontrivial solution and $t_0$ is a zero of $y$ then there cannot exist a sequence $\{t_n\}_{n=1}^{\infty}$ of distinct zeros of $y$ such that $t_n \xrightarrow[n\to\infty]{} t_0$.

[10]