Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

PhD Admission Test for Mathematics (2019-II)

Date: 02 December 2019

Time: 1.5 hours | Total marks: 60

Name (In BLOCK letters):

Roll/Application Number:

Category (Tick the appropriate one) : GEN/OBC-NCL/EWS/SC/ST/PwD

Instructions

1. This question booklet consists of 20 questions, divided into four sections, with 5 questions in each section.

2. Each question may have more than one correct options.

3. Each question carries 3 marks.

4. An examinee will be awarded 3 marks for a totally correct answer. For the questions containing multiple correct options, 1 mark will be given for partially correct answers, provided no wrong option has been chosen in addition. In all other cases, no marks will be awarded.

5. This question-cum-answer booklet must be returned to the invigilator before leaving the examination hall.

6. Please enter your answers ONLY on this page in the space given below.

Section A		Section B		Section C		Section D	
Q. No.	Answer	Q. No.	Answer	Q. No.	Answer	Q. No.	Answer
1		6		11		16	
2		7		12		17	
3		8		13		18	
4		9		14		19	
5		10		15		20	
Marks		Marks		Marks		Marks	

Total marks obtained:

Notations and conventions

Throughout this question paper, the following notations and conventions will be adopted:

1. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the set of all integers, rationals, real numbers and complex numbers respectively.

2. For $p \in [1, \infty)$, $L^p(\mathbb{R})$ denotes the set of all measurable functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ with the property that $\int_{\mathbb{R}} |f(t)|^p \, dt < \infty$.

3. $L^\infty(\mathbb{R})$ stands for the set of all bounded measurable functions from $\mathbb{R}$ to $\mathbb{C}$.

4. $\mathbb{D}^2 := \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk in $\mathbb{C}$.

5. $\mathbb{H}^2 := \{x + iy \in \mathbb{C} : y > 0\}$ is the upper half plane in $\mathbb{C}$.

6. $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the *Riemann sphere*.

7. For $n \in \mathbb{N}$, we let $S^n := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$.

8. For $n \in \mathbb{N}$ and a field $\mathbb{K}$, $M_n(\mathbb{K})$ denotes the set of all $n \times n$ matrices with entires from $\mathbb{K}$. When $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we assume that $M_n(\mathbb{K})$ is identified with $\mathbb{K}^{n^2}$ by the following map

$$(a_{ij})_{i,j=1}^n \longmapsto (a_{11}, \cdots, a_{1n}, \cdots, a_{n1}, \cdots, a_{nn}),$$

   and thus equipped with the natural metric topology. Consequently, the subgroups $GL_n(\mathbb{K}), SL_n(\mathbb{K})$ etc. inherit the subspace topology from $M_n(\mathbb{K})$.

9. Given any commutative ring R with identity and $a \in R$, (a) denotes the principle ideal in R generated by the element a.

Section A

1. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded continuous function. Choose the correct statement(s) from the following:

   (a) The sequence $\left\{\int_{x_n}^{y_n} g(t)\,dt\right\}_{n=1}^{\infty}$ is Cauchy for any pair of Cauchy sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in $\mathbb{R}$.

   (b) The function defined by

   $$(x, y) \mapsto \int_x^y g(t)\,dt\,, \text{ for all } (x, y) \in \mathbb{R}^2,$$

   may not be differentiable at every point of $\mathbb{R}^2$.

   (c) The function defined in (1b) is continuous on $\mathbb{R}^2$ but not uniformly continuous on $\mathbb{R}^2$.

   (d) If $g(x_0) \neq 0$, then one can find open intervals I, J in $\mathbb{R}$ containing x_0 such that the set

   $$\mathcal{S} := \left\{(x, y) \in I \times J : \int_x^y g(t)\,dt = 0\right\}$$

   is the graph of some continuously differentiable function $\varphi : I \longrightarrow J$, i.e.,

   $$\mathcal{S} = \{(x, \varphi(x)) : x \in I\}.$$

2. Let f be a continuous real valued function on $\mathbb{R}$ with compact support. Pick out the correct statement(s) from below:

   (a) $f(\mathbb{R})$ is measurable.

   (b) The Lebesgue measure of $f(\mathbb{R})$ can be 0 even when f is nonconstant.

   (c) The boundary of $f^{-1}(-\infty, \alpha)$ has positive measure for at most countably many $\alpha \in \mathbb{R}$.

   (d) For every $p \in [1, \infty]$, there exists a continuous function $g : \mathbb{R} \longrightarrow \mathbb{R}$ which vanishes identically on $\mathbb{R} \setminus f(\mathbb{R})$ but $g \notin L^p(\mathbb{R})$.

3. Let $\gamma : [0, 1] \longrightarrow \mathbb{C}$ be continuously differentiable and γ^* denote its range. Assume $\gamma(0) = \gamma(1)$. Define $\eta_\gamma : \mathbb{C} \longrightarrow \mathbb{C}$ by the following

   $$\eta_\gamma(z) = \begin{cases} \displaystyle\int_\gamma \frac{dw}{w - z} & \text{if } z \in \mathbb{C} \setminus \gamma^* \\ 1 & \text{if } z \in \gamma^* \end{cases}.$$

   Find the true statement(s) from below:

   (a) The restriction of η_γ on $\mathbb{C} \setminus \gamma^*$ is locally constant.

   (b) η_γ vanishes at infinity, i.e. $\forall \varepsilon > 0$, there exists a compact subset K such that $|\eta_\gamma(z)| < \varepsilon$ holds for any $z \notin K$.

   (c) η_γ does not vanish identically on the complement of any compact subset of $\mathbb{C}$.

(d) None of the above.

4. Let f be an entire function. We define $\varphi : (0, \infty) \longrightarrow [0, \infty)$ by

$$\varphi(t) := \sup_{|z|=t} |f(z)|\,, \text{ for all } t > 0.$$

Which of the following statement(s) is/are always true?

(a) φ is bounded.

(b) φ has a zero, i.e. $\exists t_0 \in (0, \infty)$ such that $\varphi(t_0) = 0$.

(c) $\varphi(t) \xrightarrow[t\to\infty]{} 0$.

(d) φ is continuous except at countably many points in every closed and bounded interval $[a, b] \subseteq (0, \infty)$.

5. Let $\mathscr{H}$ be a Hilbert space over $\mathbb{C}$. Consider a bounded linear operator $T : \mathscr{H} \longrightarrow \mathscr{H}$ such that $||Tv|| \leq ||v||$, for every $v \in \mathscr{H}$. Denote the *adjoint* of T by T^*, which is defined by the following property:

$$\langle Tv, w\rangle = \langle v, T^*w\rangle\,, \text{ for all } v, w \in \mathscr{H}.$$

Pick the FALSE statement(s) from the following:

(a) $T^*v = v \Longrightarrow Tv = v$, where $v \in \mathscr{H}$.

(b) The converse of (5a) holds true.

(c) If T is an isometry (i.e. $||Tv|| = ||v||$, for all $v \in \mathscr{H}$) then $T^*T = I$, where I is the identity operator on $\mathscr{H}$.

(d) None of the above is true.

Section B

6. Consider the group $\mathbb{Z}/2019\,\mathbb{Z}$ with addition modulo 2019. Which of the following groups admit(s) an homomorphism onto $\mathbb{Z}/2019\,\mathbb{Z}$?

   (a) $\mathbb{Z}/26247\,\mathbb{Z}$ with respect to addition modulo 26247.

   (b) $\mathbb{Q}$ with respect to usual addition.

   (c) $\{z \in \mathbb{C} : \exists n \in \mathbb{Z} \text{ such that } z^n = 1\}$ with respect to usual multiplication of complex numbers.

   (d) $\{z \in \mathbb{C} : |z| = 1\}$ with respect to usual multiplication of complex numbers.

7. Let R be a commutative ring with identity. Let $a \in R$ be such that $a^{2019} = 0$ and u be a unit in R. Then the cardinality of the quotient ring $R/(u + a)$ is

   (a) 1

   (b) same as that of R.

   (c) 2019.

   (d) 2018.

8. Consider the subfields $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{7})$ of $\mathbb{C}$. Which of the following statements is/are true?

   (a) They are isomorphic as abelian groups.

   (b) They are isomorphic as vector spaces.

   (c) They are isomorphic as rings.

   (d) They are isomorphic as fields.

9. Let V be the subspace of the vector space of all 5×5 real symmetric matrices with the property that characteristics polynomial of each element in V is of the form $x^5 + ax^3 + bx^2 + cx + d$. Then the dimension of V is:

   (a) 15.

   (b) 14.

   (c) 10.

   (d) 12.

10. Suppose that A is a 5×5 real matrix all of whose entries are 1. Find the correct one(s) from the statements given below.

    (a) A is not diagonalizable over $\mathbb{R}$.

    (b) A is idempotent.

    (c) A is nilpotent.

    (d) The minimal polynomial and the characteristics polynomial of A are not same.

Section C

11. Let $f,\, g : X \longrightarrow Y$ be continuous maps where X is an arbitrary topological space and Y is a Hausdorff space. Find the true statement(s) from the following:

   (a) The subset $\{x \in X : f(x) = g(x)\} \subseteq X$ is closed in X.

   (b) Even if $f \neq g$, there can exist a dense subset $D \subseteq X$ such that $f(x) = g(x)$ for all $x \in D$.

   (c) If $f : X \longrightarrow Y$ is injective then X is also a Hausdorff topological space.

   (d) None of the above statements is true.

12. The function given by $z \mapsto \dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ such that $ad - bc > 0$, is a

   (a) holomorphic map from $\mathbb{D}^2$ onto itself with an holomorphic inverse.

   (b) holomorphic map from $\mathbb{H}^2$ onto itself with an holomorphic inverse.

   (c) holomorphic onto function on $\mathbb{C}$ with an holomorphic inverse.

   (d) holomorphic map from $\widehat{\mathbb{C}}$ onto itself with an holomorphic inverse.

13. Which of the following statements is/are true?

   (a) If $X \subseteq \mathbb{R}^2$ is path connected, then $\overline{X}$ is also path connected.

   (b) Let $X \subseteq \mathbb{R}$. Then X is connected if and only if X is path connected.

   (c) For $n \in \mathbb{N}$, let N and S be respectively the points $(0, \ldots, 0, 1)$ and $(0, \ldots, 0, -1)$ in $\mathbb{R}^{n+1}$. Then $S^n \setminus \{N, S\}$ is path connected.

   (d) The set $X = \{(x, y) \in \mathbb{R}^2 : xy = \pm 1, x > 0\}$ is path connected.

14. Consider the function $f : (0, \infty) \longrightarrow \mathbb{R}$ defined by $f(x) = \sin\left(\frac{1}{x}\right)$, for all $x \in (0, \infty)$. Pick the correct statement(s) from the following:

   (a) f has countably many fixed points.

   (b) For any $a > 0$, the restriction map $f|_{(a,\infty)} : (a, \infty) \longrightarrow \mathbb{R}$ has infinitely many fixed points.

   (c) The restriction map $f|_{(0,1]} : (0, 1] \longrightarrow \mathbb{R}$ has finitely many fixed points.

   (d) The restriction map $f|_{[1,\infty)} : [1, \infty) \longrightarrow \mathbb{R}$ has no fixed point.

15. Consider the following functions:

$$f : GL_n(\mathbb{C}) \longrightarrow \mathbb{C} \setminus \{0\},\ f(A) := \det A, \text{ for all } A \in GL_n(\mathbb{C});$$

and

$$g : \mathbb{R} \longrightarrow M_2(\mathbb{R}),\ g(x) := \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}, \text{ for all } x \in \mathbb{R}.$$

Choose the correct one(s) from the statements given below:

   (a) Let $\mathscr{K}$ denote the Cantor set and $GL_n(\mathscr{K})$ denote the set of all $n \times n$ invertible matrices having entries from $\mathscr{K}$. Then $f(GL_n(\mathscr{K}))$ is closed.

(b) Let $\mathscr{K}$ be as above. Then $g(\mathscr{K})$ is closed.

(c) $GL_n(\mathbb{C})$ has infinitely many closed subgroups containing $SL_n(\mathbb{C})$.

(d) All the above three statements are true.

Section D

16. If the function $K : [0,1] \times [0,1] \to \mathbb{R}$ is such that

$$u(t) = c_1 + c_2 t + \int_0^t K(t,s) f(s)\, ds$$

is the general solution to the ODE

$$\frac{d^2}{dt^2} u(t) = f(t), \; 0 < t < 1,$$

where f is continuous on $[0,1]$, then $K(t,s) =$

(a) $s - t$.

(b) $t - s$.

(c) $t(s - t)$.

(d) $s(t - s)$.

17. The differential equation

$$y = x\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$$

has more than one solutions passing through the point

(a) $(0, 1)$.

(b) $(1, 1)$.

(c) $(2, 1)$.

(d) $(2, -1)$.

18. Let $u \in C^2(\mathbb{R} \times [0, \infty))$ solves the initial value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 & \text{for all } (x,t) \in \mathbb{R} \times [0,\infty) \\ u(x,0) = f(x) & \text{for all } x \in \mathbb{R} \\ u_t(x,0) = g(x) & \text{for all } x \in \mathbb{R}\,, \end{cases}$$

where f and g are infinitely differentiable functions with compact supports. For $t > 0$, define

$$K(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx \text{ and } P(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx.$$

Choose the correct one(s) from the following statements.

(a) The function $K(t) + P(t)$ is a constant function of time.

(b) The function $K(t) + P(t)$ can be a non constant function of time.

(c) The function $K(t) + P(t)$ is always continuous.

(d) The function $K(t) + P(t)$ is a polynomial of degree 3.

19. Let $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a C^1 function (i.e., both partial derivatives are continuous). Consider the following problem:

$$\begin{cases} u_t(x,t) + u_x(x,t) = 0 & \text{for all } \ (x,t) \in \mathbb{R}^2 \\ u(x,x) = 1 & \text{for all } x \in \mathbb{R}\,. \end{cases}$$

Which of the following statements is/are correct?

(a) The above problem has unique solution.

(b) The above problem has infinitely many solutions.

(c) There exists a solution u of the above problem such that $u(1,0) = 5$.

(d) The above problem has at most finitely many solutions.

20. Consider the following function

$$\phi : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R},\ \phi(r) := \frac{1}{2r} \int_{1-r}^{1+r} u(s)\, ds, \text{ for all } r \in \mathbb{R} \setminus \{0\}\,;$$

where $u'' = 0$ on $\mathbb{R}$ with $u(1) \neq 0$. Then

(a) $\phi'(r) = 0$ for all $r \in \mathbb{R} \setminus \{0\}$.

(b) $\phi(1) = u(1)$.

(c) $\lim\limits_{r \to 0} \phi(r) = u(0)$.

(d) ϕ is an odd function.

Space for rough work

Space for rough work

Space for rough work

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